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Appendix D: Notes on Anisotropy in linear elasticity
Supplementary material for the course of Solid Mechanics

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APPENDIX D: Definition of Anisotropy in Linear Elasticity

A material is *linearly elastic* if its applied stress field $\sigma(\boldsymbol{x})$ is related to the resulting strain field $\varepsilon(\boldsymbol{x})$ by a linear relation or the generalized Hooke law,

$$\sigma(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x})\varepsilon(\boldsymbol{x}) \quad , \quad \sigma_{kl}(\boldsymbol{x}) = C_{klmn}(\boldsymbol{x})\varepsilon_{mn}(\boldsymbol{x}) \qquad (k,l,m,n = 1,2,3) . \tag{D.1}$$

Here $\boldsymbol{C}$ is a fourth-order tensor and called the *stiffness tensor*. The usual summation convention is applied here. In mechanics, second-order tensors are familiar but fourth-order ones are much less known. Nevertheless, they follow the general rule of orthogonal transformation between coordinates, i.e.,

$$C'_{klmn}(\boldsymbol{x}) = c_{ki}c_{lj}c_{mp}c_{nq}C_{ijpq}(\boldsymbol{x}) \qquad (k,l,m,n,i,j,p,q = 1,2,3) . \tag{D.2}$$

If $\boldsymbol{C}$ does not depend on $\boldsymbol{x}$, the corresponding linearly elastic material is *homogeneous*; otherwise, it is *inhomogeneous*[1]. The nine equations relate nine components of stress to the nine corresponding strain components. Thus, there are $3^4 = 81$ elastic constants at most, corresponding to the indices taking the values 1, 2, and 3. The first equation is,

$$\begin{aligned}\sigma_{11} &= C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} \\ &\quad + C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23} \\ &\quad + C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33} \ .\end{aligned}$$

Next, let us examine the basic properties of the stiffness tensor $\boldsymbol{C}$. Note that $\boldsymbol{C}$ is a linear transformation from the *symmetric* strain tensor space to the *symmetric* stress tensor space. As a result, the matrix components C_{ijkl} of $\boldsymbol{C}$, called *elastic constants*, satisfy the following relations,

$$C_{klmn} = C_{lkmn} = C_{klnm} \ . \tag{D.3}$$

The symmetries expressed by these two equalities are usually referred to as the *minor symmetries* of $\boldsymbol{C}$. These two symmetries reduce the number of elastic constant to 36.

In linear elasticity, we adopt two important hypotheses:

1. For an adiabatic or isothermal process, there exists a *strain energy density function W,* which is also a potential for the stresses,

[1] For simplicity, the dependence on $\boldsymbol{x}$ will not be written out whenever it is evident or irrelevant

$$\boldsymbol{\sigma} = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad , \quad \sigma_{ij} = \frac{\partial W(\boldsymbol{\varepsilon}_{ij})}{\partial \boldsymbol{\varepsilon}_{ij}} . \tag{D.4a}$$

2. The stability hypothesis which states that the stiffness tensor is *positive definite*, i.e.,

$$\varepsilon : \boldsymbol{C}\varepsilon = C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0, \quad \forall \quad \varepsilon_{ij} \neq 0 \tag{D.4b}$$

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{C\varepsilon} \quad ; \quad W(\boldsymbol{\varepsilon}_{kl}) = \frac{1}{2}C_{ijkl}\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{kl} . \tag{D.4c}$$

These two hypotheses amount to the following results,

$$\begin{aligned} \frac{\partial W}{\partial \varepsilon_{11}} &= \sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + ... + C_{1132}\varepsilon_{32} \\ \frac{\partial W}{\partial \varepsilon_{22}} &= \sigma_{22} = C_{2211}\varepsilon_{11} + C_{2222}\varepsilon_{22} + ... + C_{2232}\varepsilon_{32} \end{aligned} \tag{D.5a}$$

$$\frac{\partial^2 W}{\partial \varepsilon_{11}\partial \varepsilon_{22}} = C_{1122} = C_{2211} \text{ or in general } \frac{\partial^2 W}{\partial \varepsilon_{kl}\partial \varepsilon_{mn}} = C_{klmn} = C_{mnkl} . \tag{D.5b}$$

Accounting for these results, in the general anisotropic case the number of independent elastic constants is reduced to *twenty one*. In matrix form relation, (D.1) is,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix} . \tag{D.6}$$

With the symmetry conditions given above, the matrix of the elastic constant is symmetric and called the *stiffness matrix*.

The stress-strain relations given by (D.6) can also be presented in an inverted form as follows,

$$\boldsymbol{\varepsilon} = \boldsymbol{S\sigma} \quad , \quad \varepsilon_{kl} = S_{klmn}\sigma_{mn} \tag{D.7}$$

where S_{klmn} are constants for a homogeneous material and are the elements of a fourth order tensor $\boldsymbol{S} = \boldsymbol{C}^{-1}$. This tensor is called *compliance tensor* and has the same symmetry properties as $\boldsymbol{C}$. Relations (D.7) can be derived from the scalar function,

$$W^*(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{S\sigma} = \frac{1}{2} S_{ijkl}\sigma_{ij}\sigma_{kl} \qquad \text{(D.8a)}$$

and

$$\boldsymbol{\varepsilon}_{ij} = \frac{\partial W^*(\sigma_{ij})}{\partial \sigma_{ij}} \quad , \quad \boldsymbol{\varepsilon} = \frac{\partial W^*(\sigma)}{\partial \sigma} \qquad \text{(D.8b)}$$

where W^* is called *stress energy density function*.

In matrix form the generalized Hook's law using the compliance is also written as follows,

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1112} & S_{1113} & S_{1123} \\ S_{2211} & S_{2222} & S_{2233} & S_{2212} & S_{2213} & S_{2223} \\ S_{3311} & S_{3322} & S_{3333} & S_{3312} & S_{3313} & S_{3323} \\ S_{1211} & S_{1222} & S_{1233} & S_{1212} & S_{1213} & S_{1223} \\ S_{1311} & S_{1322} & S_{1333} & S_{1312} & S_{1313} & S_{1323} \\ S_{2311} & S_{2322} & S_{2333} & S_{2312} & S_{2313} & S_{2323} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix} \qquad \text{(D.9)}$$

with $S_{klmn} = S_{mnkl}$.

Overall, due to the symmetries of stress and stress tensors as well as the symmetry due to the energy expressed in (D.5b), we can write the following equalities for the stiffness or compliance elements of the matrices in the constitutive relations,

$$C_{klmn} = C_{lkmn} = C_{klnm} = C_{nmkl}$$

With these symmetries, the matrix elements can be simplified as shown in the table below,

tensor notation	11	22	33	23,32	31,13	12,21
matrix notation+	1	2	3	4	5	6

+This notation is used in the literature for layered composite materials

This simplification is widely used in mechanics of composites mechanics materials and will be implemented in some applications shown later.

Basic Cases of Elastic Symmetry

Materials that obey the generalized Hook's law are in general different. With respect to their elastic properties, all engineering materials can be divided into *isotropic* and *anisotropic*. An isotropic elastic material is one in which the elastic properties are the same in all directions drawn through a point. Depending on its structure, a material could be isotropic or anisotropic.

The symmetry of an elastic material depends upon the symmetry of its structure. The relationship between the structural and elastic symmetry for crystals was established according to F. Neumann's principle: "*the symmetry of the elastic properties of a solid contains that of its crystallographic structure*".

If there is symmetry of the elastic properties (i. e., elastic symmetry) in an anisotropic material, the corresponding generalized Hook's law is simpler since some of the coefficients of C_{klmn} are zero or related by linear relationships. These simplifications can be derived by applying the method summarized next.

Let the material be referred to a coordinate system $Ox_1x_2x_3$ and to a second one $Ox_1'x_2'x_3'$ symmetric with respect to the first one, the symmetry being the same as that observed in the structure. Since the directions of the respective axes x_1', x_2', x_3' and x_1, x_2, x_3 are equivalent as regards the elastic properties, the generalized Hook's law must be the same in the two coordinate systems. After writing these equations in both these systems, we transform to either of them expressing the elastic constants, say x_1', x_2', x_3' in terms of x_1, x_2, x_3. On comparing the resulting similar equations, we find relationships between C_{klmn} and C_{klmn}' (or S_{klmn} and S_{klmn}'). Below we present the well-known material symmetries. Note that every case is discussed with respect to the material principal axes.

Symmetry with respect to one plane

The material that exhibits elastic symmetry with respect to one plane is called *monoclinic***.**

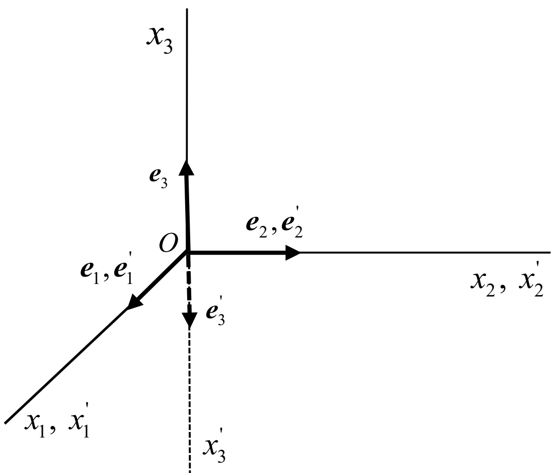


Fig. D1: Monoclinic symmetry.

The symmetry is expressed by the requirement that the elastic constants do not change under a change from the system $Ox_1x_2x_3$ to the system $Ox_1^{'}x_2^{'}x_3^{'}$ (Fig. D1). Imposing the above requirement on the matrices (D.6) and (D.9), the number of the elastic constants reduces to *thirteen*. Namely,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & C_{3312} & 0 & 0 \\ C_{1112} & C_{2212} & C_{3312} & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & C_{1323} \\ 0 & 0 & 0 & 0 & C_{1323} & C_{2323} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}. \qquad \text{(D.10)}$$

Interestingly, more solids belong to the monoclinic system than to any other one. Typical examples are natural materials like**,** kaolin (a clay material) and muscovite (or mica).

Symmetry with respect to two orthogonal planes.

A material that exhibits elastic symmetry with respect to two orthogonal planes is called an *orthotropic material.*

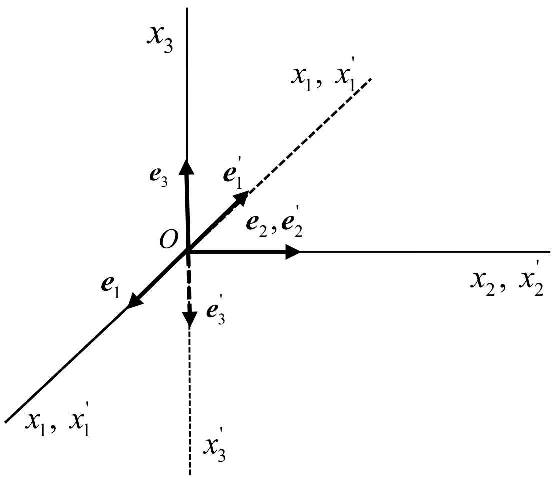


Fig. D2: Orthotropic symmetry.

Similarly, the symmetry is expressed by the requirement that the elastic constants remain the same under a change from the system $Ox_1x_2x_3$ to the system $Ox_1^{'}x_2^{'}x_3^{'}$ (Fig. D2). The number of the elastic constants is reduced to *nine* and the stress-strain relation become,

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{2323} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}. \tag{D.11}
$$

In this category, we find materials like wood, layered polymer composites, several crystals and rolled metals.

Symmetry with respect to one axis.

A material that possesses an axis of symmetry, in the sense that all rays at right angles to this axis are equivalent, is called *transversely isotropic*. That is, the elastic properties should be the same in all systems of axes shown in Fig. D3.

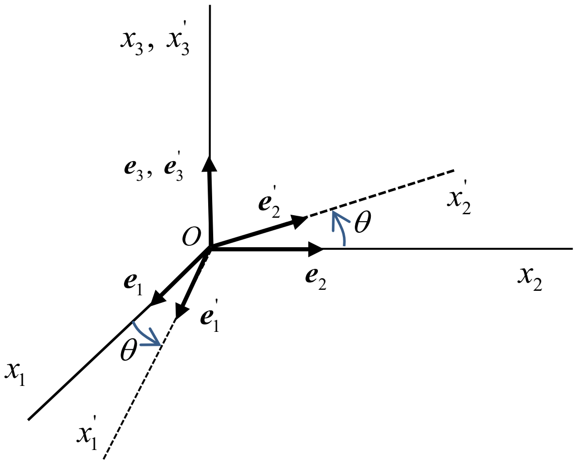


Fig. D3: Transverse isotropic symmetry.

The number of independent coefficients is reduced to *five* and the stress-strain relation is given by,

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0 \\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{1111}-C_{1122}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1313} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}. \tag{D.12}
$$

An isotropic material possesses elastic symmetry that is independent of the orientation of the axes. Following the same procedure, we find that, the number of the elastic constants is reduced to *two* and presented in the following way,

$$C_{1111} = \lambda + 2\mu \;\; ; \;\; C_{1122} = \lambda \;\; ; \;\; C_{1212} = (C_{1111} - C_{1212}) = 2\mu \qquad \text{(D.13)}$$

where λ and μ are the so-called Lamé constants, related to Young's modulus E and Poisson's ratio ν by ,

$$\lambda = E\nu / (1+\nu)(1-2\nu), \quad \mu = E / 2(1+\nu).$$

Based on the last identification the stress-strain relation for an isotropic linear material (D.1), in matrix form, is given by,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}. \qquad \text{(D.14a)}$$

In index notation they are,

$$\sigma_{kl} = \lambda \varepsilon_{pp} \delta_{kl} + 2\mu \varepsilon_{kl} \qquad \text{(D.14b)}$$

with their inverse given by,

$$\varepsilon_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda+2\mu)} \sigma_{kk} + \frac{1}{2\mu} \sigma_{ij}\,. \qquad \text{(D.14c)}$$

For an isotropic material, it is also possible to express C in (D.1) as a fourth order isotropic tensor. Note that an isotropic tensor in Euclidian space is a tensor whose components remain the same in any rectangular Cartesian system related by orthogonal transformation.

In this case we have,

$$C_{klmn} = \lambda \delta_{kl} \delta_{mn} + \mu(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) \qquad \text{(D.15a)}$$

and

$$\sigma_{kl} = C_{klmn}\varepsilon_{mn} = \left[\lambda\delta_{kl}\delta_{mn} + \mu(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\right]\varepsilon_{mn}$$
$$= \lambda\varepsilon_{mm}\delta_{kl} + 2\mu\varepsilon_{kl} \, . \qquad \text{(D.15b)}$$

For a linear elastic isotropic material, the strain energy density function W in (D.4a) takes the form,

$$W(\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} \, . \qquad \text{(D.16)}$$

This function can be differentiated with respect to strains to obtain the stresses following the steps below,

$$\sigma_{pq} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{pq}} = \frac{1}{2}\lambda\left(\frac{\partial \varepsilon_{ii}}{\partial \varepsilon_{pq}}\varepsilon_{kk} + \varepsilon_{ii}\frac{\partial \varepsilon_{kk}}{\partial \varepsilon_{pq}}\right) + 2\mu\varepsilon_{ij}\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{pq}}$$

$$\sigma_{pq} = \frac{1}{2}\lambda\left(\delta_{ip}\delta_{iq}\varepsilon_{kk} + \varepsilon_{ii}\delta_{kp}\delta_{kq}\right) + 2\mu\varepsilon_{ij}\delta_{ip}\delta_{jq}$$
$$= \frac{1}{2}\lambda\left(2\delta_{pq}\varepsilon_{kk}\right) + 2\mu\varepsilon_{pq} = \lambda\delta_{pq}\varepsilon_{kk} + 2\mu\varepsilon_{pq}$$

which is expression (D.14b). This important result defines the strain energy density (D.16) as a stress tensor potential, which can be established thermodynamically.

Applications to composites

High strength composites are made of layers of laminae, i.e., laminates with the reinforcing fibers along one (unidirectional composites) or several directions (multidirectional composites). Structural components from such materials are strong and light but the relationships between the stiffness and engineering constants are not very simple. To establish the constitutive relations, it is better to relate the compliance coefficients with engineering constants and then invert the matrix of the compliance to obtain the stiffness. In the subsequent paragraphs, we present steps to follow in order to express such relations in the principal material and rotated coordinate system.

A typical lamina, or an orthotropic material element, is made of continuous carbon or glass fibers and epoxy matrix as shown in Fig. D4 with respect to the principal material axes,

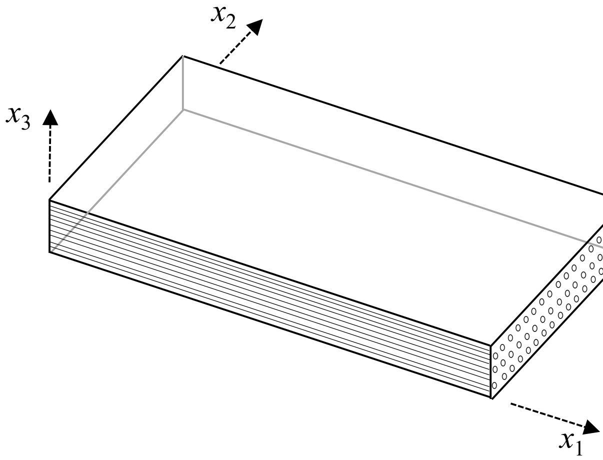


Fig. D4: Typical lamina (unidirectional reinforcement).

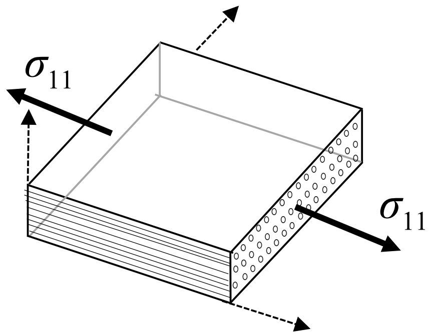
The constitutive relations, i.e. strain-stress, using the compliance can be obtained by inverting the stiffness matrix (D.11),

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{1122} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{1133} & S_{2233} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{2323} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix}. \tag{D.17}$$

Comparing it with (D.11), we see the same structure with the nine independent constants. Also, it is interesting to make the following remarks:

1. There is no coupling between normal stresses and shear strains
2. There is no coupling between shear stresses and normal strains
3. There is no coupling between a shear stress acting on one plane and a shear stress on a different plane.

It is important to note that by changing the reference system of coordinates, the number of independent constant does not but the matrix is fully populated. To express the elements of the matrix (D.17) with the engineering constants, we consider the following simple loading cases:

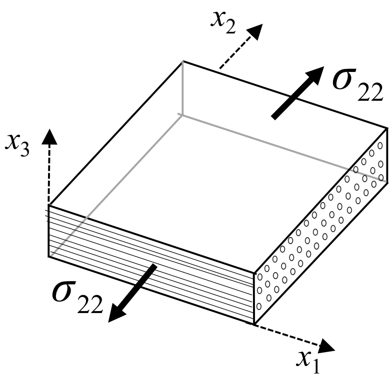


1. Longitudinal tension along direction 1

From (D.17) we have,

$$\varepsilon_{11} = S_{1111}\sigma_{11} = \frac{1}{E_1}\sigma_{11} \quad ; \quad \varepsilon_{22} = S_{1122}\sigma_{11} = -\frac{\nu_{12}}{E_1}\sigma_{11}$$

$$\varepsilon_{33} = S_{1133}\sigma_{11} = -\frac{\nu_{13}}{E_1}\sigma_{11} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

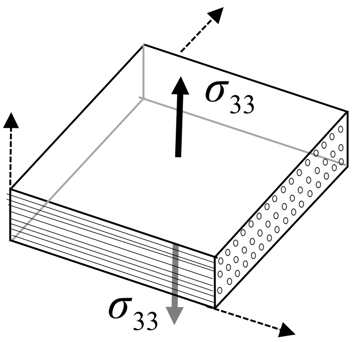


2. Transverse tension (in-plane) along direction 2

From (D.17) we have,

$$\varepsilon_{11} = S_{1122}\sigma_{22} = -\frac{\nu_{21}}{E_2}\sigma_{22} \quad ; \quad \varepsilon_{22} = S_{2222}\sigma_{22} = \frac{1}{E_2}\sigma_{22}$$

$$\varepsilon_{33} = S_{1133}\sigma_{33} = -\frac{\nu_{23}}{E_2}\sigma_{22} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

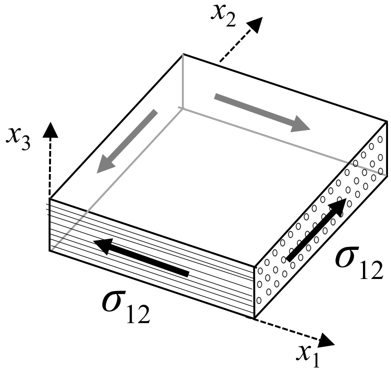


3. Transverse tension (out of plane) along direction

From (D.17) we have,

$$\varepsilon_{11} = S_{1133}\sigma_{33} = -\frac{\nu_{31}}{E_3}\sigma_{33} \quad ; \quad \varepsilon_{22} = S_{2233}\sigma_{33} = -\frac{\nu_{32}}{E_3}\sigma_{33}$$

$$\varepsilon_{33} = S_{1133}\sigma_{33} = \frac{1}{E_3}\sigma_{33} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

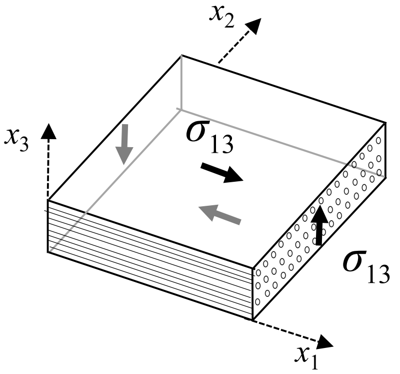


4. In plane shear on the 1-2 plane

From (D.17) we have,

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$$

$$\varepsilon_{12} = \frac{1}{2G_{12}}\sigma_{12} \quad ; \quad \varepsilon_{13} = \varepsilon_{23} = 0$$

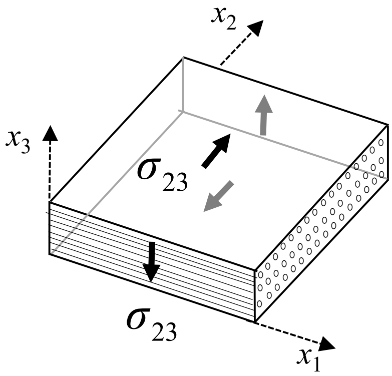


5. Out of plane shear on the 1-3 plane

From (D.17) we have,

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$$

$$\varepsilon_{13} = \frac{1}{2G_{13}}\sigma_{13} \quad ; \quad \varepsilon_{12} = \varepsilon_{23} = 0$$



6. Out of plane shear on the 2-3 plane

From (D.17) we have,

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$$

$$\varepsilon_{23} = \frac{1}{2G_{23}}\sigma_{23} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = 0$$

Thus, relations expressed in (D.17) are,

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix} . \qquad \text{(D.18)}$$

Note that the shear components in (D.18) can be simplified. However, they are left for consistency, and correspondence/comparison with (D.17). In the following paragraphs, we will simplify these elements when it is appropriate. ,

The symmetry requirements on the above compliance matrix result in the following relations,

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} \quad ; \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3} \quad ; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \tag{D.19a}$$

or in general

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad or \quad \frac{\nu_{ij}}{\nu_{ji}} = \frac{E_i}{E_j}. \tag{D.19b}$$

It is easy to see here the nine independent elastic constants, i.e. E_1, E_2, E_3, G_{12}, G_{13}, G_{23}, and $\nu_{21}, \nu_{12}, \nu_{13}, \nu_{31}, \nu_{23}, \nu_{32}$ related by the three symmetry relations (D.19).

Structural panels made of layered composites are relatively thin and subjected to in-plane loads. Thus, they are in a state to plane stress. It is interesting to deduce the strain-stress as well as stress-strain relations in such cases.

We consider an orthotropic material as it is the most common composite material used in structural panels.

For plane stress problems, the non-zero stress components are σ_{11}, σ_{22}, $\sigma_{12} = \sigma_{21} \neq 0$ and σ_{13}, σ_{23}, $\sigma_{33} = 0$ as shown in Fig. D5.

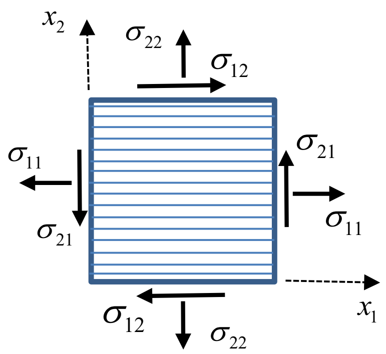


Fig. D5: Plane stress state with reference to the principal material axe

For this plane case the full matrix relation (D.17) reduces to,

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & 0 \\ S_{1122} & S_{2222} & 0 \\ 0 & 0 & S_{1212} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{pmatrix}. \tag{D.20a}$$

And in terms of the engineering constants,

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_2} & 0 \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{4G_{12}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ 2\sigma_{12} \end{pmatrix}. \tag{D.20b}$$

Notice here that the number of independent elastic constants reduces to *four* $E_1, E_2, G_{12}, v_{21}, v_{12}$ with the symmetry $v_{21} / E_2 = v_{12} / E_1$.

In mechanics of composites, the constitutive relations shown earlier needs to be simplified because of tensorial nature of stress, strain and the compliance that makes transformation between coordinate systems complex. Thus, the following convention is adopted in the literature. In this summary we present the case of a plane stress, which is the most common state of stress encountered in applications of composite materials as structural components. In the adopted simplification, using the engineering strains $\gamma_{12} = 2\varepsilon_{12}$, the stresses and strains with reference to the principal material axes (x_1, x_2) are indicated by,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} \quad ; \quad \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}.$$

In terms of the compliance we have,

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix}. \tag{D.21a}$$

From (D.20b) the elastic constants are,

$$S_{11} = \frac{1}{E_1};\ S_{22} = \frac{1}{E_2};\ S_{12} = S_{21} = -\frac{v_{21}}{E_2} = -\frac{v_{12}}{E_1};\ S_{66} = \frac{1}{G_{12}} \tag{D.21b}$$

In terms of stiffness, relation (D.21a) can be inverted to obtain the constitutive relations as follows,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} \tag{D.22a}$$

with,

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \quad ; \quad Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}$$
$$Q_{12} = Q_{21} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \quad ; \quad Q_{66} = G_{12} \quad . \tag{D.22b}$$

With reference to another system of axes (x, y), they are indicated as,

$$\begin{pmatrix} \sigma_x \\ \sigma_x \\ \sigma_{xy} \end{pmatrix} \text{ and } \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{D.23}$$

with the corresponding constitutive relations,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} \tag{D.24}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{ys} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}. \tag{D.25}$$

Note here that both matrices are full which demonstrates a coupling between normal and shear stresses. However, the matrices are symmetric and the number of independent constants remains the same, i.e. four independent elastic constants.

Changes in the coordinate system

Very often the lamina principal axes ((x_1, x_2) do not coincide with the loading axes (x, y)). Then the stress-strain relation referred to the principal axes can be expressed in terms of the relations with reference to (x, y). Note that the transformation of stresses and strains in Cartesian coordinates are independent of material properties. That is, they are the same if the material is isotropic or anisotropic. Fig. D.6 shows the stress state at a point in an orthotropic material with respect to (x_1, x_2) and (x, y). The stresses σ_{11}, σ_{22}, σ_{12} can be expressed in terms of the standard equations found in the transformation of a second order tensor (or Morh's circle), i.e.,

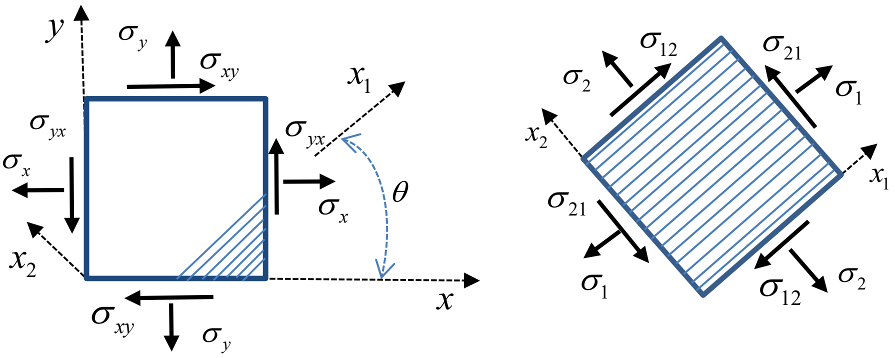


Fig. D6: State of stress of an element in plane stress.

$$\begin{aligned}
\sigma_1 &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\sigma_{xy}\cos\theta\sin\theta \\
\sigma_2 &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\sigma_{xy}\cos\theta\sin\theta \\
\sigma_{12} &= \sigma_{21} = -(\sigma_x - \sigma_y)\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

For simplicity, we introduce $c = \cos\theta,\ s = \sin\theta$. It is often convenient in the mechanics of composites materials, to express these transformation equations in matrix form as follows,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = [T] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} . \qquad \text{(D.26)}$$

As for the strains, the same transformation rules apply and the form of the transformation relation is the same,

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \varepsilon_{xy}\cos\theta\sin\theta \\
\varepsilon_2 &= \varepsilon_x \sin^2\theta + \sigma_y \cos^2\theta - \varepsilon_{xy}\cos\theta\sin\theta \\
\varepsilon_{12} &= -(\varepsilon_x - \varepsilon_y)\cos\theta\sin\theta + \varepsilon_{xy}(\cos^2\theta - \sin^2\theta).
\end{aligned}
\Rightarrow \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{pmatrix} = \begin{pmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -cs & cs & c^2 - s^2 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{pmatrix} = [T] \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{pmatrix}$$

Since we introduced the engineering strain earlier, $\gamma_{12} = 2\varepsilon_{12},\ \gamma_{xy} = 2\varepsilon_{xy}$, a factor of 2 appears in the third equation. Thus, matrix $[T]$ is modified accordingly,

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = [T'] \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}. \qquad \text{(D.27)}$$

In inverted form, (D.27) becomes,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \left[T'\right]^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}. \tag{D.28}$$

The next step is to express the elements of compliance matrix in (D.24) in terms of the elements with reference to the principal material axes (x_1, x_2). They can be obtained by transformation of the strain-stress relations following the steps below,

$$\begin{aligned} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} &= \left[T'\right]^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \left[T'\right]^{-1} \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} \\ &= \left[T'\right]^{-1} \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \left[T\right] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}. \end{aligned} \tag{D.29}$$

Comparing it with (D.24), we have,

$$\begin{pmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{pmatrix} = \left[T'\right]^{-1} \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \left[T\right]. \tag{D.30}$$

Explicitly, the constants are related as follows,

$$\begin{aligned} S_{xx} &= S_{11}c^4 + S_{22}s^4 + \left(2S_{12} + S_{66}\right)c^2s^2 \\ S_{yy} &= S_{11}s^4 + S_{22}c^4 + \left(2S_{12} + S_{66}\right)c^2s^2 \\ S_{ss} &= \left(4S_{11} + 4S_{22} - 8S_{12} - 2S_{66}\right)c^2s^2 + S_{66}\left(c^4 + s^4\right) \\ S_{xy} &= \left(S_{11} + S_{22} - S_{66}\right)c^2s^2 + S_{12}\left(c^4 + s^4\right) \\ S_{xs} &= \left(2S_{11} - 2S_{12} - S_{66}\right)c^3s - \left(2S_{22} - 2S_{12} - S_{66}\right)cs^3 \\ S_{ys} &= \left(2S_{11} - 2S_{12} - S_{66}\right)cs^3 - \left(2S_{22} - 2S_{12} - S_{66}\right)c^3s. \end{aligned} \tag{D.31}$$

Using similar steps, we can express the stiffness matrix in (D.25) with reference to an (x, y) coordinate system,

$$
\begin{aligned}
Q_{xx} &= Q_{11}c^4 + Q_{22}s^4 + \left(2Q_{12} + 4Q_{66}\right)c^2s^2 \\
Q_{yy} &= Q_{11}s^4 + Q_{22}c^4 + \left(2Q_{12} + 4Q_{66}\right)c^2s^2 \\
Q_{ss} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)c^2s^2 + Q_{66}\left(c^4 + s^4\right) \\
Q_{xy} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)c^2s^2 + Q_{12}\left(c^4 + s^4\right) \\
Q_{xs} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)c^3s - \left(Q_{22} - Q_{12} - 2Q_{66}\right)cs^3 \\
Q_{ys} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)cs^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)c^3s.
\end{aligned}
\tag{D.32}
$$

It is also of interest and useful in composites mechanics to express the engineering constants (i.e. moduli and Poisson's ratios) with reference to a coordinate system rotated at an angle theta with respect to the principal material coordinates. For a better understanding and links to the applied load, we can imagine a series of elementary tests, similar to the ones used to construct the compliance matrix (D.18). Thus, we consider a small square of a unidirectional composite whose reinforcing fibers are at an angle θ with the principal materials axes as shown in Fig. D7. Note here that we have a full matrix.

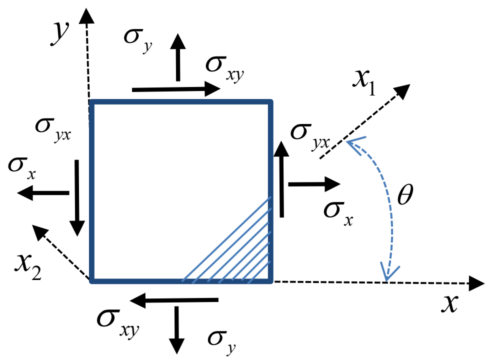


Fig. D7: Element of a composite plate loaded along axes at an angle to the principal material system.

We apply the stresses $\sigma_x, \sigma_x, \sigma_{xy} = \sigma_{yx}$, one at a time, to obtain in each case the following strain stress relations,

(a). apply $\sigma_x \quad \Rightarrow \quad \varepsilon_x = \dfrac{1}{E_x}\sigma_x, \;\; \varepsilon_y = -\dfrac{v_{xy}}{E_x}\sigma_x, \;\; \gamma_{xy} = \dfrac{\eta_{xs}}{E_x}\sigma_x.$

(b). apply $\sigma_y \quad \Rightarrow \quad \varepsilon_x = -\dfrac{v_{yx}}{E_y}\sigma_y, \;\; \varepsilon_y = \dfrac{1}{E_y}\sigma_y, \;\; \gamma_{xy} = \dfrac{\eta_{ys}}{E_y}\sigma_y.$

(c). apply $\sigma_{xy} \quad \Rightarrow \quad \varepsilon_x = \dfrac{\eta_{sx}}{G_{xy}}\sigma_{xy}, \;\; \varepsilon_y = \dfrac{\eta_{sy}}{G_{xy}}\sigma_{xy}, \;\; \gamma_{xy} = \dfrac{1}{G_{xy}}\sigma_{xy}$ (or pure shear).

Here the coefficients $\eta_{xs}, \eta_{ys}, \eta_{sx}, \eta_{sy}$ are the shear coupling coefficients. In matrix form we have,

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{v_{yx}}{E_y} & \frac{\eta_{sx}}{G_{xy}} \\ -\frac{v_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{sy}}{G_{xy}} \\ \frac{\eta_{xs}}{E_x} & \frac{\eta_{ys}}{E_y} & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}. \tag{D.33}$$

These elastic constants are related by the following symmetries,

$$\frac{v_{xy}}{E_x} = \frac{v_{yx}}{E_y} \quad ; \quad \frac{\eta_{xs}}{E_x} = \frac{\eta_{sx}}{G_{xy}} \quad ; \quad \frac{\eta_{ys}}{E_y} = \frac{\eta_{sy}}{G_{xy}}. \tag{D.34}$$

A comparison with the matrix form of strain – stress relations (D.24),

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} \tag{D.24bis}$$

results in the following,

$$S_{xx} = \frac{1}{E_x}; \quad S_{yy} = \frac{1}{E_y}; \quad S_{ss} = \frac{1}{G_{xy}} \quad . \tag{D.35a}$$

$$S_{xy} = S_{yx} = -\frac{v_{xy}}{E_x} = -\frac{v_{yx}}{E_y} \tag{D.35b}$$

$$S_{xs} = S_{sx} = \frac{\eta_{xs}}{E_x} = \frac{\eta_{sx}}{G_{xy}} \tag{D.35c}$$

$$S_{ys} = S_{sy} = \frac{\eta_{ys}}{E_y} = \frac{\eta_{sy}}{G_{xy}} \tag{D.35d}$$

We can also express the moduli, Poisson's ratios and shear coupling coefficients in terms of $S_{ij}, \ (i, j = x, y, s)$ appropriate combination of (D.35). To obtain the stiffness matrix, we can invert the matrix in (D.33).

Next we express the variation of the elastic constants as a function of θ, angle of the loading direction with respect to the material principal direction (Fig. D7). This can be achieved by combining (D.35) and (D.31). The resulting elastic expressions for the constants are useful because they allow for the examination of the influence of loading direction (with respect to the material principal directions) on the strain-stress response of a lamina.

A typical case is shown below for the Young's modulus along orientation defined by axis x. We start with (D.31), replace parameters S_{ij} with the corresponding expressions in (D.35) and the symmetry conditions (D.19a) to obtain,

$$\begin{aligned}
\frac{1}{E_x} &= \frac{1}{E_1}c^4 + \frac{1}{E_2}s^4 - 2\frac{\nu_{12}}{E_1}c^2s^2 + \frac{1}{G_{12}}c^2s^2 \\
\frac{1}{E_x} &= \frac{1}{E_1}c^4 + \frac{1}{E_2}s^4 - \frac{\nu_{12}}{E_1}c^2s^2 - \frac{\nu_{21}}{E_2}c^2s^2 + \frac{1}{G_{12}}c^2s^2 \\
\frac{1}{E_x} &= \left(\frac{1}{E_1}c^4 - \frac{\nu_{12}}{E_1}c^2s^2\right) + \left(\frac{1}{E_2}s^4 - \frac{\nu_{21}}{E_2}c^2s^2\right) + \frac{1}{G_{12}}c^2s^2
\end{aligned} \qquad \text{(D.36)}$$

$$\Rightarrow \frac{1}{E_x} = \frac{c^2}{E_1}\left(c^2 - \nu_{12}s^2\right) + \frac{s^2}{E_2}\left(s^2 - \nu_{21}c^2\right) + \frac{1}{G_{12}}c^2s^2. \qquad \text{(D.37)}$$

A similar procedure is followed to express the other elastic constants in terms of the orientation angle.

$$\begin{aligned}
\frac{1}{E_x} &= \frac{c^2}{E_1}\left(c^2 - \nu_{12}s^2\right) + \frac{s^2}{E_2}\left(s^2 - \nu_{21}c^2\right) + \frac{1}{G_{12}}c^2s^2 \\
\frac{1}{E_y} &= \frac{s^2}{E_1}\left(s^2 - \nu_{12}c^2\right) + \frac{c^2}{E_2}\left(c^2 - \nu_{21}s^2\right) + \frac{1}{G_{12}}c^2s^2 \\
\frac{1}{G_{xy}} &= \frac{4s^2c^2}{E_1}\left(1 + \nu_{12}\right) + \frac{4s^2c^2}{E_2}\left(1 + \nu_{21}\right) + \frac{(c^2 - s^2)^2}{G_{12}} \\
\frac{\nu_{xy}}{E_x} &= \frac{\nu_{yx}}{E_y} = \frac{c^2}{E_1}\left(\nu_{12}c^2 - s^2\right) + \frac{s^2}{E_2}\left(\nu_{21}s^2 - c^2\right) + \frac{1}{G_{12}}c^2s^2 \\
\frac{\eta_{xs}}{E_x} &= \frac{\eta_{sx}}{G_{xy}} = \frac{2c^3s}{E_1}\left(1 + \nu_{12}\right) - \frac{2cs^3}{E_2}\left(1 + \nu_{21}\right) - \frac{cs}{G_{12}}\left(c^2 - s^2\right) \\
\frac{\eta_{ys}}{E_y} &= \frac{\eta_{sy}}{G_{xy}} = \frac{2cs^3}{E_1}\left(1 + \nu_{12}\right) - \frac{2c^3s}{E_2}\left(1 + \nu_{21}\right) + \frac{cs}{G_{12}}\left(c^2 - s^2\right)
\end{aligned} \qquad \text{(D.38)}$$

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